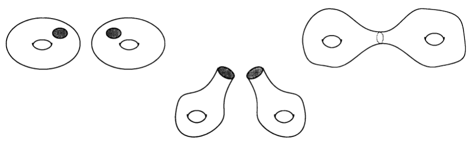
TOPOLOGY - III, EXERCISE SHEET 3

The aim of this exercise sheet is to classify all compact surfaces up to homeomorphism and compute their (simplicial) homology. By a compact surface we mean a compact connected real manifold of dimension 2 without boundary.

We recall the definition of the connected sum of two surfaces:

Definition 3.1: Let M and N be surfaces. The connected sum of M and N denoted by $M\#N$ is the surface obtained from M and N as follows. First remove open disks from M and from N. This results in two new surfaces $M - D$ and $N - D'$ with boundaries homeomorphic to S^1. Then $M\#N$ is the surface obtained by gluing $M - D$ and $N - D$ along the boundaries.



Exercise 1. (easy)
Let M be a compact surface. Show that $M\#S^2$ is homeomorphic to M.

Now we consider the notion of a triangulation of a compact surface. By a triangle we mean a 2-simplex.

Definition 3.2: A triangulated surface is a $\Delta-$complex comprising of the data of a set of triangles $\{T_i\}_{i\in I}$ with glueing data obeying the following conditions:

(1) Given two triangles T_i, T_j; Either T_i, T_j are disjoint or they must be identified along a single edge or identified at a single vertex.
(2) Every edge of a Triangle is identified with exactly one other edge.
(3) If some k triangles meet at a vertex, then they can be labelled $S_1, ..., S_k$ so that S_i shares an edge with S_{i+1} and S_k shares an edge with S_1.

We will assume the following theorem of Rado (1925) which will make the proof of the classification theorem purely combinatorial.

Theorem 3.3: Let X be a compact surface, then X is a triangulated surface.

Assuming Rado's Theorem, consider the following exercise:

Exercise 2. (easy)

(1) Show that if X is a compact surface then it can be triangulated by a finite number of triangles.
(2) Using connectedness of X show that the triangles in the triangulation of X can be enumerated as $T_1, ..., T_n$ such that T_i shares and edge with T_{i+1} for $i = 1, ..., n-1$.

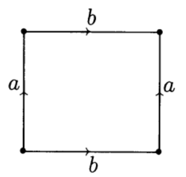
Exercise 3. (medium) Find a triangulation for:
(1) S^2
(2) $\mathbb{T}^2$
(3) $\mathbb{RP}^2$

Now we state our main theorem. Which we will prove through a series of steps and exercises.

Theorem 3.4: Let X be a compact surface, then X is homeomorphic to either S^1, a connected sum of k copies of $\mathbb{T}^2$ for some $k > 0$ or a connected sum of k copies of $\mathbb{RP}^2$ for some $k > 0$.

Step 1: Building a planar model of the surface X

Definition: A planar model for a compact surface X is a polygon with oriented edges, which are labelled in pairs such that X can be obtained by gluing the edges of the polygon with the same labelling along the prescribed orientation. See for example a planar model for $\mathbb{T}^2$:



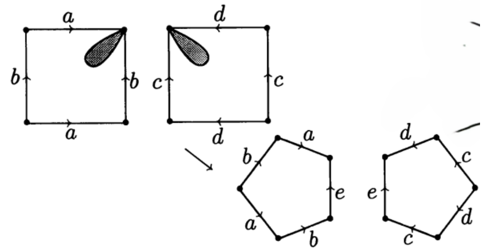
The above planar diagram can be denoted by the notation $aba^{-1}b^{-1}$. This is obtained by tracing the edges of the planar diagram in a clockwise manner and writing the edge labelling or it's inverse depending on the orientation.

Exercise 4. (easy)
(1) Using Exercise 2.2, show that any compact surface has a planar diagram.
(2) Find planar Diagrams for the following surfaces:
  (a) S^2
  (b) $\mathbb{T}^2$
  (c) $\mathbb{RP}^2$
(3) What surfaces do the following planar diagrams correspond to?
  (a) $aba^{-1}b^{-1}$.
  (b) aa.

(c) $aba^{-1}b$.

Exercise 5. (medium)

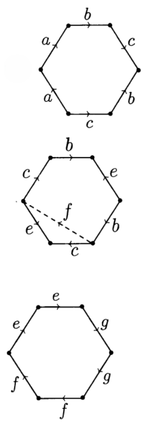
(1) Suppose X and Y are compact surfaces, obtain a planar diagram for $X\#Y$ from the planar diagrams of X and Y. **Hint:** Delete open discs as in the following example:



(2) Show that if w_1 is a planar diagram of X and w_2 is a planar diagram for Y, then w_1w_2 is a plananr diagram for $X\#Y$.
(3) Show that $a_1b_1a_1^{-1}b_1^{-1}...a_nb_na_n^{-1}b_n^{-1}$ is a planar diagram for the connected sum of n copies of $\mathbb{T}^2$.
(4) Show that $a_1a_1a_2a_2...a_na_n$ is a planar diagram for the connected sum of n copies of $\mathbb{RP}^2$.
(5) Show that after the identification of edges are made, the surfaces in part (3) and (4) have only one vertex arising from the planar diagram

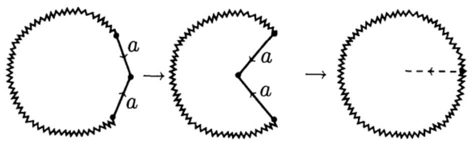
Our strategy will be to start with a surface, form it's planar diagram (Exercise 4.1) and perform operations consisting of cutting and glueing to form a planar diagram as in Exercise 5.1 and Exercise 5.2. We will need the following exercise:

Exercise 6. (hard)
By manipulating planar diagrams, show that $\mathbb{T}^2\#\mathbb{RP}^2$ is homeomorphic to $\mathbb{RP}^2\#\mathbb{RP}^2\#\mathbb{RP}^2$. **Hint:**Try to get the following intermediate diagram:



Step 2: Eliminate adjacent opposing pairs of edges

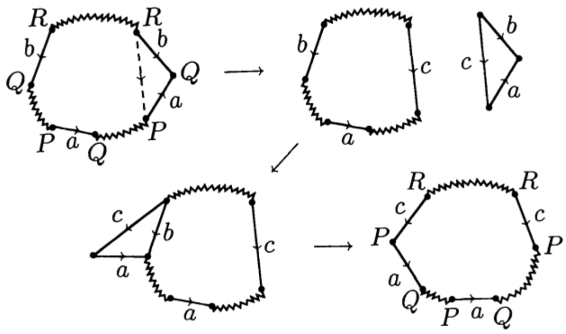
We can get rid of such a pair by considering it as an internal edge of the triangulation as in the proof of Exercise 4.1 following the step as in the diagram:



So we can essentially cancel a and a^{-1} whenever they appear.

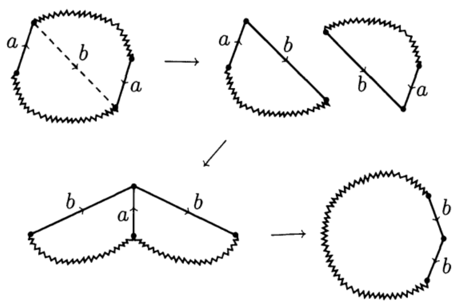
Step 3: Eliminate all but one vertex

We perform a series of operations such that the quotient of the resulting planar diagram has only one vertex. We follow the algorithm as in the following example where we eliminate an instance of the vertex Q in favour of the vertex P.



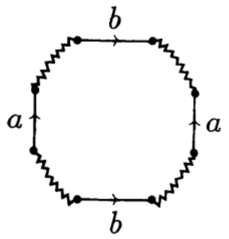
Step 4: Collect twisted pairs of edges

If we have a twisted pair of edges in the planar diagram for X, that is the planar diagram for X looks like $w_1aw_2aw_3$, then we can carry out an operation to make these edges adjacent as depicted in the following diagram:



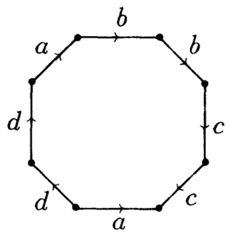
Step 5: Collect pairs of pairs of opposing edges

The title of this step refers to a situation as follows:



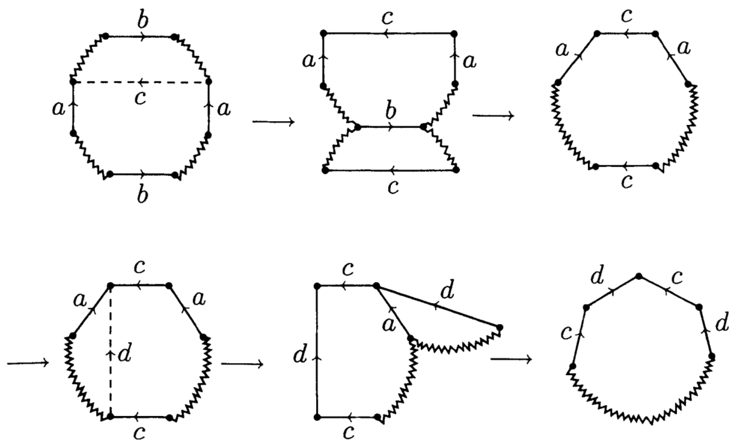
Exercise 7. (Hard)

Show that if after Step 4 is carried out, the planar diagram of X must look like the one in Exercise 5.4 or there must exist pairs of pairs of edges such that the planar diagram is as in the above diagram ie. $aw_1bw_2a^{-1}w_3b^{-1}$. In other words such a situation is not possible after Step 4 where there is only one pair of opposing edges:



Hint: Count the number of vertices in the above diagram.

We can then collect such pairs of pairs of opposing edges as in the following diagram:

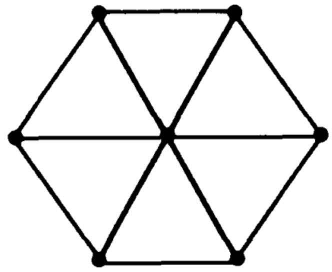


Step 6: Conclusion

Exercise 8. (easy)

Conclude Theorem 3.4 from Step 4, Exercise 5.3, Step 5, Exercise 5.4 and Exercise 6.

Exercise 9. (medium)
Let X be a compact surface, we give X a $\Delta-$complex structure by introducing a vertex in the interior of the planar diagram and thereby dividing it into triangles as follows:



Calculate the simplicial homology groups of $\mathbb{T}^{2\#n}$ and $\mathbb{RP}^{2\#n}$, where $X^{\#n}$ is notation for the connected sum of n copies of X. Conclude that the first simplicial homology group $H^1(X, \mathbb{Z})$ determines the compact surface X.